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ON SOME CYCLE RELATED ABSOLUTE MEAN GRACEFUL GRAPHS

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Abstract: For a graph G of size q, an absolute mean graceful labeling g of a graph G is an injective mapping from the set of vertices of G to the set $\{0, \pm 1, \pm 2, ..., \pm q\}$ such that when each edge vw is assigned the label $\lceil \frac{|g(v)-g(w)|}{2} \rceil$, the resulting edge labels are $1, 2, ..., q$. If a graph G admits this labeling, then it is called an absolute mean graceful graph. In this paper, we construct some absolute mean graceful graphs of higher order obtained from cycles using various graph operations.

Keywords and Phrases: Absolute mean graceful labeling, Cycle, Cyclic snakes, Switching of a vertex, Duplication.

2020 Mathematics Subject Classification: 05C78, 05C76, 05C38.

1. Introduction

By a graph $G = (V(G), E(G))$ we mean a finite, simple, undirected and connected graph with p vertices (i.e. order of G is p) and q edges (i.e. size of G is q).

For a given graph G, any mapping which assigns values to the edges or vertices or both under certain condition(s) is known as graph labeling. The concept of graph labelings is introduced by Rosa [15]. Almost 3600 publications have been published in the intervening years that examine more than 350 graph labeling techniques. Labeled graphs have different practical applications, which can be seen in [6, 16, 18, 20]. β-valuation was initially introduced by Rosa [15]. Golomb

[9] further referred to this labeling as graceful labeling. Several type of labelings are introduced and that can be found in [8]. The notion of an absolute mean graceful labeling has been introduced by Kaneria and Chudasama [11].

Definition 1.1. [11] *A function g is said to be an absolute mean graceful labeling if it is one-to-one function from $V(G)$ to the set $\{0, \pm 1, \pm 2, ..., \pm q\}$ such that when each edge vw is assigned the label $\left\lceil \frac{|g(v)-g(w)|}{2} \right\rceil$, the resulting edge labels are $1, 2, ..., q$. If G admits this labeling, it is called an absolute mean graceful graph.*

Various graph families are proved absolute mean graceful in [1, 11] and in [2, 3, 7, 12, 13] several absolute mean graceful graphs have been constructed using different graph operations.

In this paper, we construct some absolute mean graceful graphs of higher order obtained from n- cycles using various graph operations. For any undefined terminology regarding graph theory we follow West [19].

Definition 1.2. [10] *The block-cutpoint graph of a graph G is a bipartite graph, denoted by $bc(G)$, in which one partite set consists of the cut vertices of G, and the other consists of a vertex b_i for each block B_i of G. We include xb_i as an edge of $bc(G)$ if and only if $x \in B_i$.*

Note that $bc(G)$ is a tree whenever G is a connected graph.

Definition 1.3. [5] *rC_n-snake is a connected graph in which r blocks are isomorphic to the n- cycle C_n and the block-cutpoint graph is a path graph.*

Badr [4] generalized the Definition 1.3 which is defined as follows.

Definition 1.4. [4] *The family of graphs consisting of r copies of C_n of C_n with two non-adjacent vertices in common where every copy has k copies of C_n and the block-cutpoint graph is a path graph is denoted by $(k, r)C_n$-snake.*

Definition 1.5. [4] *$(k, r)C_n$-snake is said to be linear, if the block-cut-vertex graph has the property that the distance between any two consecutive cut-vertices is $\left\lfloor \frac{n}{2} \right\rfloor$.*

Definition 1.6. [8] *A vertex switching graph $\widetilde{G}$ is the graph constructed from G by deleting all edges incident to x and introducing edges connecting to every non-adjacent vertex of x in G.*

Definition 1.7. [17] *Duplication of a vertex x_j by a new edge $y'y''$ in a graph G produces a new graph H such that $N(y') = \{x_j, y''\}$ and $N(y'') = \{x_j, y'\}$.*

Definition 1.8. [14] *Let $x_1, x_2, \ldots, x_n$ be the consecutive vertices of P_n. The irregular triangular snake $I(T_n)$ is constructed by joining x_j and x_{j+2} to a new vertex y_j, where $j = 1, 2, \ldots, (n-2)$.*

2. Main Results

Theorem 2.1. *All the linear $(k, r)C_4$-snakes are absolute mean graceful.*
Proof. Let G be a linear $(k, r)C_4$-snake graph. Let $x_1, x_2, \ldots, x_{r+1}$ are vertices of block-cutpoint graph. Let y_{ij} vertices are adjacent to x_i and x_{i+1}, where $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, 2k$. Clearly, order of G is $p = (2k+1)r + 1$ and size of G is $q = 4kr$.
Define vertex labeling $g : V(G) \to \{0, \pm 1, \pm 2, \ldots, \pm 4kr\}$ by

$$g(x_i) = q - 4k(i-1); \quad i = 1, 2, \ldots, r+1,$$
$$g(y_{ij}) = -q + 4k(i-1) + 2j - 2; \quad i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, 2k.$$

The vertex labeling g defined above is one-to-one and induced edge labels are $1, 2, \ldots, 4kr$. Hence, all the linear $(k, r)C_4$-snakes are absolute mean graceful.

Illustration 2.1. Absolute mean graceful labeling of $(3, 2)C_4$-snake is shown in the Figure 1.

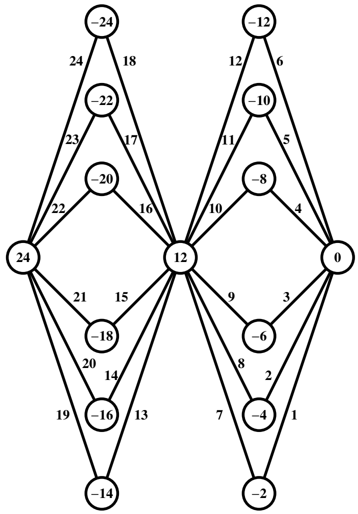

Figure 1: $(3, 2)C_4$-snake and its absolute mean graceful labeling

Theorem 2.2. *All the linear $(k, r)C_8$-snakes are absolute mean graceful.*
Proof. Let G be a linear $(k, r)C_8$-snake graph. Let $x_1, x_2, \ldots, x_{r+1}$ are vertices of block-cutpoint graph. Let y_{ij} vertices are adjacent to x_i and x_{i+1}, where $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, 2k$. Let $z_{(2i-1)j}$ vertices are adjacent to x_i and y_{ij}, where $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, 2k$. Let $z_{(2i)j}$ vertices are adjacent to x_{i+1} and y_{ij}, where $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, 2k$. Clearly, order of G is $p = (6k+1)r + 1$ and size of G is $q = 8kr$.

Define vertex labeling $g : V(G) \to \{0, \pm 1, \pm 2, \ldots, \pm 8kr\}$ by

$$g(x_i) = q - 8k(i-1); \quad i = 1, 2, \ldots, r+1,$$
$$g(y_{ij}) = q - 8ki + 4j - 2; \quad i = 1, 2, \ldots, r, \quad j = 1, 2, \ldots, 2k,$$
$$g(z_{ij}) = -q + 4k(i-1) + 2j - 2; \quad i = 1, 2, \ldots, 2r, \quad j = 1, 2, \ldots, 2k.$$

The vertex labeling g defined above is one-to-one and induced edge labels are $1, 2, \ldots, 8kr$. Hence, all the linear $(k, r)C_8$-snakes are absolute mean graceful.

Illustration 2.2. Absolute mean graceful labeling of $(2, 2)C_8$-snake is shown in the Figure 2.

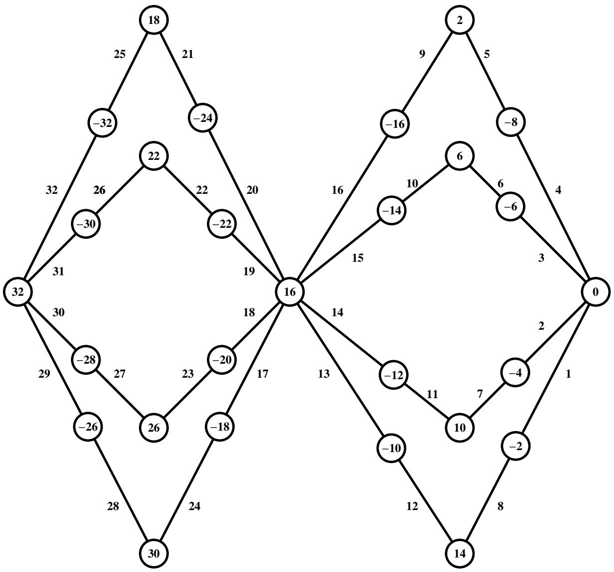

Figure 2: $(2, 2)C_8$-snake and its absolute mean graceful labeling

Theorem 2.3. *$\widetilde{C_n}$ is an absolute mean graceful graph for all $n \geq 3$.*
Proof. Let $y_1, y_2, \ldots, y_n$ be consecutive vertices of n-cycle C_n. Without loss of generality let the switched vertex be y_1. Clearly, order of $\widetilde{C_n}$ is $p = n$ and size of $\widetilde{C_n}$ is $q = 2n - 5$.
We consider following two cases to define vertex labeling
$g : V(\widetilde{C_n}) \to \{0, \pm 1, \pm 2, \ldots, \pm(2n-5)\}$

Case-1: For even n

<u>Subcase-1</u>: For $n = 4$

$\widetilde{C_4} \cong K_{1,3}$ which is proved in [11].

Subcase-2: For $n \geq 6$

$$g(y_j) = \begin{cases} q, & \text{if } j = 1; \\ -1, & \text{if } j = 2; \\ (-1)^j(q - 2(j-3)), & \text{if } j = 3, 4, \ldots, n-1; \\ 1, & \text{if } j = n. \end{cases}$$

Case-2: For odd n

Subcase-1: For $n = 3$

$\widetilde{C_3} \cong K_2 \cup K_1$, which is a disconnected graph. Since switched vertex is y_1, it is correspond to K_1 and y_2, y_3 are correspond to K_2.

$$g(y_1) = 0,$$
$$g(y_2) = 1,$$
$$g(y_3) = -1.$$

Subcase-2: For $n \geq 5$

$$g(y_j) = \begin{cases} q, & \text{if } j = 1; \\ 1, & \text{if } j = 2; \\ (-1)^j(q - 2(j-3)), & \text{if } j = 3, 4, \ldots, n-1; \\ -1, & \text{if } j = n. \end{cases}$$

The vertex labeling g defined above in both cases is one-to-one and induced edge labels are $1, 2, \ldots, (2n-5)$. Hence $\widetilde{C_n}$ is an absolute mean graceful graph for all $n \geq 3$.

Illustration 2.3. Absolute mean graceful labeling of $\widetilde{C_8}$ is shown in the Figure 3.

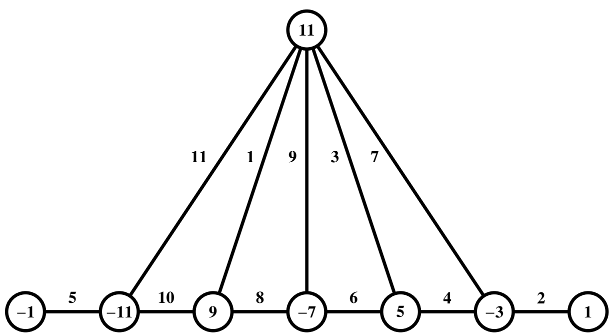

Figure 3: $\widetilde{C_8}$ and its absolute mean graceful labeling

Remark 1. *It should be noted in Theorem 2.3 that* $\widetilde{C_3}$ *preserves its absolute mean graceful labeling even if the vertex labels defined in Theorem 2.3 are interchanged.*

Remark 2. *We have seen in Theorem 2.3 that* $\widetilde{C_3} \cong K_2 \cup K_1$ *is absolute mean graceful. Now, one can observe that the graph* $K_2 \cup 2K_1$ *is not absolute mean graceful as there is no injective map from the set of vertices of* $K_2 \cup 2K_1$ *to the set* $\{0, 1, -1\}$*. Similarly,* $K_2 \cup mK_1$ *is not absolute mean graceful for* $m > 2$*.*

Theorem 2.4. *The graph obtained by duplicating each pendant vertex of* $\widetilde{C_n}$ *by a new edge is an absolute mean graceful graph for all* $n \geq 3$*.*

Proof. Let $y_1, y_2, \ldots, y_n$ be consecutive vertices of n-cycle C_n. Without loss of generality let the switched vertex be y_1. Then y_2 and y_n are pendant vertices. Now, we duplicate y_2 and y_n by edges x_2x_2' and x_nx_n', respectively. Let G be the required graph, then order of G is $p = n + 4$ and size of G is $q = 2n + 1$ $(n \neq 4)$. We consider following two cases to define vertex labeling $g : V(G) \to \{0, \pm 1, \pm 2, \ldots, \pm(2n + 1)\}$

Case-1: For odd n

Subcase-1: For $n = 3$

Note that, the graph is disconnected. Since the switched vertex is y_1, so it is an isolated vertex and y_2, y_3 are pendant vertices.

$$\begin{aligned}
g(y_1) &= 0,\\
g(y_2) &= 7,\\
g(y_3) &= -6,\\
g(x_2) &= -5,\\
g(x_2') &= -3,\\
g(x_3) &= 2,\\
g(x_3') &= -1.
\end{aligned}$$

Subcase-2: For $n = 5$

$$\begin{aligned}
g(y_1) &= -11,\\
g(y_2) &= -6,\\
g(y_3) &= 11,
\end{aligned}$$

$$\begin{aligned} g(y_4) &= -9, \\ g(y_5) &= 7, \\ g(x_2) &= 6, \\ g(x_2') &= -2, \\ g(x_5) &= -7, \\ g(x_5') &= 2. \end{aligned}$$

<u>Subcase-3</u>: For $n \geq 7$

$$g(y_j) = \begin{cases} -q, & \text{if } j = 1; \\ -7, & \text{if } j = 2; \\ (-1)^{(j+1)}(q - 2(j-3)), & \text{if } j = 3, 4, \ldots, n-1; \\ q - 10, & \text{if } j = n. \end{cases}$$

$$\begin{aligned} g(x_2) &= q - 5, \\ g(x_2') &= g(x_2) - 8, \\ g(x_n) &= g(y_n) - 15, \\ g(x_n') &= g(x_n) + 4. \end{aligned}$$

Case-2: For even n

<u>Subcase-1</u>: For $n = 4$

Since $\widetilde{C_4} \cong K_{1,3}$, there are 3 pendant vertices, namely y_1, y_2, and y_4. So, we have to duplicate y_1, y_2 and y_4 by edges x_1x_1', x_2x_2' and x_4x_4', respectively to obtain the required graph G. Note that the order of G is $p = 10$, and the size of G is $q = 12$.

$$\begin{aligned} g(y_1) &= -12, \\ g(y_2) &= -6, \\ g(y_3) &= 12, \\ g(y_4) &= 2, \\ g(x_1) &= 10, \\ g(x_1') &= 8, \\ g(x_2) &= 9, \\ g(x_2') &= 6, \\ g(x_4) &= -5, \\ g(x_4') &= -11. \end{aligned}$$

Subcase-2: For $n = 6$

$$\begin{aligned} g(y_1) &= -13, \\ g(y_2) &= -5, \\ g(y_3) &= 13, \\ g(y_4) &= -11, \\ g(y_5) &= 9, \\ g(y_6) &= -7, \\ g(x_2) &= 6, \\ g(x_2') &= 3, \\ g(x_6) &= 7, \\ g(x_6') &= -2. \end{aligned}$$

Subcase-3: For $n \geq 8$

$$g(y_j) = \begin{cases} -q, & \text{if } j = 1; \\ -5, & \text{if } j = 2; \\ (-1)^{(j+1)}(q - 2(j-3)), & \text{if } j = 3, 4, \ldots, n-1; \\ -q + 13, & \text{if } j = n. \end{cases}$$

$$\begin{aligned} g(x_2) &= q - 5, \\ g(x_2') &= g(x_2) - 8, \\ g(x_n) &= g(y_n) + 15, \\ g(x_n') &= g(x_n) + 4. \end{aligned}$$

The vertex labeling g defined above is one-to-one in both cases and induced edge labels are $1, 2, \ldots, (2n+1)$. Hence, the graph obtained by duplicating each pendant vertex of $\widetilde{C_n}$ by a new edge is an absolute mean graceful graph for all $n \geq 3$.

Illustration 2.4. Absolute mean graceful labeling of the graph obtained by duplicating each pendant vertex of $\widetilde{C_{10}}$ by a new edge is shown in the Figure 4.

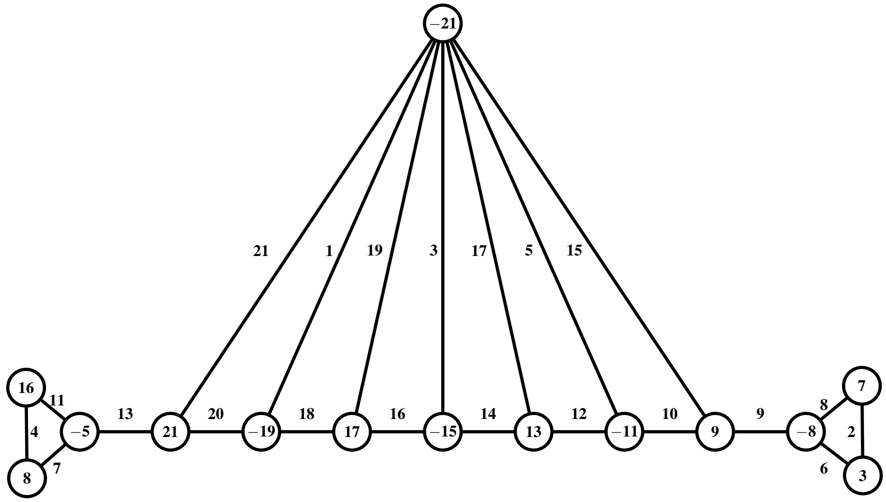

Figure 4: The graph obtained by duplicating each pendant vertex of $\widetilde{C_{10}}$ by a new edge and its absolute mean graceful labeling

Theorem 2.5. *$I(T_n)$ is an absolute mean graceful graph for all $n \geq 3$.*
Proof. Let $x_1, x_2, \ldots, x_n$ are the vertices of P_n. To obtain $I(T_n)$ join x_j and x_{j+2} to a new vertex y_j, where $j = 1, 2, \ldots, (n-2)$. Clearly, order of $I(T_n)$ is $p = 2n-2$ and size of $I(T_n)$ is $q = 3n-5$.
We consider following two cases to define vertex labeling $g : V(I(T_n)) \rightarrow \{0, \pm 1, \pm 2, \ldots, \pm(3n-5)\}$

Case-1: For odd n

<u>Subcase-1</u>: For $n = 3$

$I(T_3) \cong C_4$ which is proved in [11].

<u>Subcase-2</u>: For $n \geq 5$

$$g(x_j) = \begin{cases} -q + j - 1, & \text{if } j = 1, 3, \ldots, n; \\ q - (j-2), & \text{if } j = 2, 4, \ldots, n-1. \end{cases}$$

$$g(y_j) = \begin{cases} n - 3, & \text{if } j = 1; \\ q - 4n + 13, & \text{if } j = 2; \\ g(y_{j-2}) - 6, & \text{if } j = 3, 5, \ldots, n-2; \\ g(y_{j-2}) + 6, & \text{if } j = 4, 6, \ldots, n-3. \end{cases}$$

Case-2: For even n

Subcase-1: For $n = 4$

$$g(x_j) = \begin{cases} -q + j - 1, & \text{if } j = 1, 3; \\ q - (j - 2), & \text{if } j = 2, 4. \end{cases}$$

$$g(y_j) = \begin{cases} n - 3, & \text{if } j = 1; \\ q - 4n + 13, & \text{if } j = 2. \end{cases}$$

Subcase-2: For $n \geq 6$

$$g(x_j) = \begin{cases} -q + j - 1, & \text{if } j = 1, 3, \ldots, n - 1; \\ q - (j - 2), & \text{if } j = 2, 4, \ldots, n. \end{cases}$$

$$g(y_j) = \begin{cases} n - 3, & \text{if } j = 1; \\ q - 4n + 13, & \text{if } j = 2; \\ g(y_{j-2}) - 6, & \text{if } j = 3, 5, \ldots, n - 3; \\ g(y_{j-2}) + 6, & \text{if } j = 4, 6, \ldots, n - 2. \end{cases}$$

The vertex labeling g defined above is one-to-one in both cases and induced edge labels are $1, 2, \ldots, (3n - 5)$. Hence, $I(T_n)$ is an absolute mean graceful graph for all $n \geq 3$.

Illustration 2.5. Absolute mean graceful labeling of $I(T_9)$ is shown in the Figure 5.

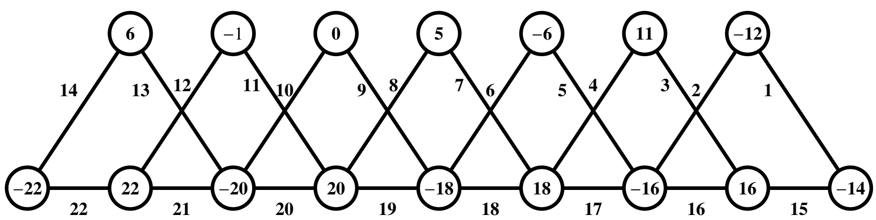

Figure 5: $I(T_9)$ and its absolute mean graceful labeling

3. Conclusion

In this paper, we have investigated some new absolute mean graceful graphs of higher order obtained from cycles using various graph operations. To investigate absolute mean graceful labeling for new graph families and establish similar results for other graph labelings is an open area of research.

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References

[1] Akbari P. Z., Kaneria V. J. and Parmar N. A., Absolute Mean Graceful Labeling of Jewel Graph and Jelly Fish Graph, International Journal of Mathematics Trends and Technology, 68(1), (2022), 86-93.

[2] Akbari P. Z., Kaneria V. J. and Parmar N. A., Some Absolute Mean Graceful Graphs in the Context of Barycentric Subdivision, International Journal of Mathematics Trends and Technology, 68(5), (2022), 51-60.

[3] Akbari P. Z., Shah J. M. and Kaneria V. J., On Absolute Mean Graceful Labeling of Disjoint Union of Some Graphs, International Journal of Mathematics and its Applications, 12(2), (2024), 61-71.

[4] Badr E. M., On Graceful Labeling of the Generalization of Cyclic Snakes, Journal of Discrete Mathematical Sciences and Cryptography, 18(6), (2015), 773-783.

[5] Barrientos C., Graceful Labelings of Cyclic Snakes, Ars Combinatoria, 60 (2001), 85-96.

[6] Bloom G. S. and Golomb S. W., Applications of numbered undirected graphs, In: Proceedings of IEEE, Vol. 65, (1977), 562-570.

[7] Chudasama H. P. and Kaneria V. J., Duplication of some graph elements and absolute mean graceful labeling, International Journal of Mathematics Trends and Technology, 65(4), (2019), 107-115.

[8] Gallian J. A., A Dynamic Survey of Graph Labeling, The Electronic Journal of Combinatorics, 26, (2023), #DS6.

[9] Golomb S. W., How to Number a Graph, In: Read R. C. (ed.) Graph Theory and Computing, Academic Press, New York, (1972), 23-37.

[10] Harary F. and Prins G., The block-cutpoint tree of a graph, Publicationes Mathematicae Debrecen, 13, (1966), 103-107.

[11] Kaneria V. J. and Chudasama H. P., Absolute mean graceful labeling in various graphs, International Journal of Mathematics and its Applications, 5(4-E), (2017), 723-726.

[12] Kaneria V. J., Chudasama H. P. and Andharia P. P., Absolute mean graceful labeling in path union of various graphs, Mathematical Journal of Interdisciplinary Sciences, 7(1), (2018), 51-56.

[13] Kaneria V. J. and Shah J. M., Absolute Mean Graceful Labeling in the Context of m-Splitting and Degree Splitting Graphs, South East Asian Journal of Mathematics and Mathematical Sciences, 19(3), (2023), 359-370.

[14] Ponraj R. and Narayanan S. Sathish, Difference Cordiality of Some Snake Graphs, Journal of Applied Mathematics and Informatics, 32(3-4), (2014), 377-387.

[15] Rosa A., On Certain Valuations of the Vertices of a Graph, In: Theory of Graphs (International Symposium, Rome, July 1966), Gordon and Breach, New York and Dunod Paris, (1967), 349-355.

[16] Sridevi R., Krishnaveni K. and Navaneethakrishnan S., A novel watershed image segmentation technique using graceful labeling, International Journal of Mathematics and Soft Computing, 3(1), (2013), 69-78.

[17] Vaidya S. K. and Barasara C. M., Product Cordial Graphs in the Context of Some Graph Operations, International Journal of Mathematics and Scientific Computing, 1(2), (2011), 1-6.

[18] Wang H., Xu J. and Yao B., Exploring New Cryptographical Construction of Complex Network Data, In: 2016 IEEE First International Conference on Data Science in Cyberspace (DSC), (2016), 155-160.

[19] West D. B., Introduction to Graph Theory, Prentice-Hall of India, New Delhi, (2006).

[20] Yegnanarayanan V. and Vaidhyanathan P., Some interesting applications of graph labellings, Journal of Mathematical and Computational Science, 2(5), (2012), 1522-1531.